Gary Froelich

Modeling Soft-Drink Packaging

MATHEMATICAL MODELING is the process of describing real-world phenomena in mathematical terms, obtaining mathematical results from the mathematical description, then interpreting and evaluating the mathematical results in the real-world situation. A simple diagram, shown in **figure 1**, depicts the process.

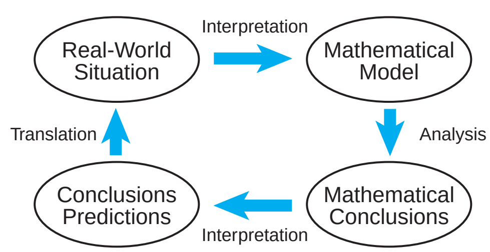


Fig. 1
The process of mathematical modeling

Capturing all aspects of a situation is difficult

The mathematical-modeling process can also be described as a series of steps. For beginners, the number of steps should be minimal:

1. Identifying a real-world problem
2. Identifying important factors and representing those factors in mathematical terms
3. Using mathematical analysis to obtain mathematical results
4. Interpreting and evaluating mathematical results as they affect the real-world problem

Students who are new to mathematical modeling can better appreciate the diagram or the list of steps after they have experienced the process. Wait until the activity has been completed to formally discuss the process.

Emphasize the idea of process to your students. Encourage them to use *model* and *modeling* as verbs. Although a collection of equations or a computer program that implements the equations is often called a *model,* a model is really more—it includes assumptions about the phenomena being modeled, for example, and the modeler must consider those assumptions if the mathematical results are unacceptable.

In typical practice, at least a portion of the modeling process is repeated because initial results are inadequate or have become ineffective over time. Failure can occur because the process missed some important aspect of reality—a common occurrence, since capturing all aspects of a situation is difficult.

Modelers strive for simplicity. Students tend to resist simplification: questions that begin “But what about . . .” are common. Emphasize simplification, and ask students to delay discussing secondary concerns until they obtain results. Those concerns may need to be addressed when the mathematical results are tested against reality. Although the modeling process is only a shadow of reality, it produces results: mathematical modeling has helped take us to the moon, helped win wars, and saved lives by predicting natural disasters. Modeling has

Gary Froelich, g.froelich@pop.comap.com, is director of high school projects at the Consortium for Mathematics and Its Applications (COMAP). His interests include mathematical modeling and educational materials that develop mathematical ideas in the context of contemporary applications

*Edited by **A. Darien Lauten,** dlauten@nh.ultranet.com, Rivier College, Nashua, NH 03061*

This section is designed to provide in reproducible formats mathematics activities appropriate for students in grades 7–12. This material may be reproduced by classroom teachers for use in their own classes. Readers who have developed successful classroom activities are encouraged to submit manuscripts, in a format similar to the “Activities” already published, to the senior journal editor for review. Of particular interest are activities focusing on the Council’s curriculum standards, its expanded concept of basic skills, problem solving and applications, and the uses of calculators and computers.

Write to NCTM, attention: infocentral, or send e-mail to infocentral@nctm.org, for the catalog of educational materials, which lists compilations of “Activities” in bound form.—Ed.

helped us accomplish these goals because it attempts to apply mathematics to only the essential elements of a situation.

Modeling can be theory driven or data driven. The results of data-driven modeling are often confirmed theoretically; conversely, theory-driven results are often confirmed by data collection—either in the real world or through simulation. For example, analyzing data on stopping distance of automobiles indicates that a quadratic function describes the relationship between automobile speed and stopping distance. A theoretical analysis shows that stopping distance has a linear component (reaction distance) and a component (braking distance) that varies with the square of the automobile's speed (Giordano, Weir, and Fox 1997, pp. 103–9).

In this activity, students conduct theory-driven modeling to improve the efficiency of secondary soft-drink packaging. The activity is adapted from *Mathematics: Modeling Our World: Course 2,* Unit 4: "The Right Stuff." Mathematics: Modeling Our World is a four-year high school mathematics program developed by the Consortium for Mathematics and Its Applications (COMAP) through a grant from the National Science Foundation.

The modeling process is accessible to students of various abilities. A good model is usually the product of more than just mathematical knowledge—insight and creativity are also important. COMAP conducts an annual High School Mathematical Contest in Modeling (HiMCM), in which teams of students research a real-world problem and develop a mathematical model.

For information on the Mathematics: Modeling Our World program or HiMCM, call (800) 772-6627 or send an e-mail to info@pop.comap.com.

TEACHER'S GUIDE

Prerequisites

- The ability to find areas of circles, triangles, and rectangles; the formula ,

$$\frac{s^2\sqrt{3}}{4},$$

  where s is the length of a side, for the area of an equilateral triangle is useful, but $(1/2)bh$ is sufficient.
- An understanding of 30°-60° right-triangle relationships
- The ability to find surface area and volume of rectangular solids and cylinders

The preceding list of prerequisites is somewhat misleading, because portraying mathematical modeling as a process in which the modeler knows all the necessary mathematics is unrealistic. Often the modeler must research not only the context but also the mathematics. In fact, new mathematics must be developed in some situations. Thus, the teacher can use portions of this activity to encourage students to develop new mathematical ideas. For example, if the students do not know 30°-60° right-triangle relationships, needing to evaluate such packages as the triangular six-pack or the hexagonal seven-pack can serve as motivation. The activity can be paused at the appropriate time, the mathematics developed, and then the activity resumed.

New mathematics must be developed in some situations

Whether or not modelers know all the mathematics needed in a situation, they often need to research the contextual setting. Since soft drinks and their packaging are familiar to students, you may think that research is unnecessary. However, a modest amount of research realistically depicts the modeling process and can enhance students' motivation.

An Internet search on "soft drink packaging" is likely to produce interesting results. Web sites of the National Soft Drink Association (**www.nsda.org/**) and the Canadian Soft Drink Association (**www.softdrink.ca/**) are good starting points. Students are likely to discover that packaging in general is a major component of landfills; the volume and recyclability of packaging material therefore are important concerns. Students may also discover that effective use of space is important in package design. For example, soft drinks were formerly packaged in bottles that used space inefficiently; as a result, some retailers stocked soft drinks only in the summer.

If your students conduct research, have them report their findings to the class. Focus the discussion on research that relates to the use of space or to the nature of packaging material.

Although the activity, as presented here, assumes an ability to work with variables, it can easily be adapted for use with less sophisticated students. →

For example, you might give students six circles of radius 3.2 cm and have them build the various configurations by arranging the circles on a sheet of paper and then drawing the package boundary. They can calculate areas from direct measurements. Imprecise measurements and inaccurate drawings can serve as reasons to introduce an electronic drawing utility or an analytic investigation. Most students can benefit from this hands-on experience, and it is especially recommended as a first step in question 4 on **sheet 3.**

Sheet 1

Optimizing two quantities simultaneously is difficult

Sheet 1 establishes secondary soft-drink packaging as the modeling context and gives students an opportunity to flex some of the mathematical muscles that they will need. In this situation, the muscles are geometric formulas.

Since students will find only slight differences among some package designs, question 1 shows that small differences can be worth pursuing. For example, an innovation that only slightly improves the efficiency of automobile engines could be worth millions of dollars to the inventor and to the automobile industry.

The goal of improving package design is too broad, so question 2 begins to define the problem. The primary concerns of this activity are maximizing the package space used by the cans, 2(*a*), and minimizing packaging material per can, 2(*d*). In the real world, the former might concern those engaged in storing, transporting, or selling soft drinks; the latter might concern soft-drink manufacturers or those public officials responsible for landfills.

As noted in question 3, modelers have at least two reasons to use simplification. One is for convenience—a simple problem is easier to tackle than a complicated one, and the knowledge gained can help with the more complicated problem. The other reason is to eliminate relatively unimportant factors. Simplification, however, carries risks, no matter what the modeler's motivation. Therefore, question 3(*c*) anticipates problems by showing that simplification to two dimensions does not compromise the results.

Question 3 assumes that packaging material has no thickness and that no overlap occurs. These assumptions may need to be reexamined.

In question 3(*a*), students may need a hint: finding the percent of package space used by the cans requires taking the ratio of the total area of the tops of the cans to the area of the rectangular top of the package.

Question 4 makes an important point: for ease of comparison, the measure of efficiency may need modification.

After completing **sheet 1,** students should be comfortable with modeling criteria, particularly the percent of package space used by the cans and the amount of material used per can. They should realize that in the former instance, optimization means maximization; and in the latter, optimization means minimization. As they progress in the activity, students should begin to realize what every modeler knows: optimizing two quantities simultaneously is difficult.

Sheet 2

The purpose of **sheet 2** is to investigate the number of cans as a controlling factor when the modeling criterion is either maximizing the package space used or minimizing the packaging material used. Students see that the number of cans is not an important factor for the first criterion, but it is for the second.

Although this entire activity is well-suited to group work, question 1 of this sheet is especially good for this type of work. Groups can share their designs with the class by posting them in the classroom or by putting them on transparencies. Groups can be asked to defend their calculations. The previously described circular manipulatives can also be used.

Students can use a computer drawing utility to create their designs. If they do, the issue of accuracy is likely to arise. For example, measurements and calculations made with a drawing utility are not reliable if constructions are inaccurate. Transformation features of drawing utilities are often the simplest method to use to construct circles that are tangent to existing circles. If electronic sketches are accurate, measurements and calculations can substitute for a deductive investigation when such an investigation proves too difficult. **Figure 2** shows an accurately constructed "parallelopack." The measure of efficiency is the portion of package space used by the cans. Although the parallelopack is not an appropriate student design for this activi-

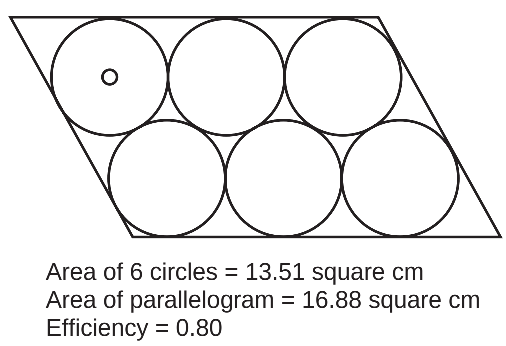


Fig. 2
An accurately constructed "parallelopack"

ty sheet, it is one that students might propose when they do **sheet 3**.

For this unit, the Geometer's Sketchpad files for several package designs can be downloaded from the COMAP Web page: **www.comap.com**. Select High School>Projects>MMOW Teacher Support>Support Library>Course 2>Unit 4. Choose either the Macintosh or Windows platform, then download the software files.

On the basis of the mathematical results that they describe in question 4, students should not conclude that the modeling process is complete. Mathematical results need interpretation and evaluation, as indicated in question 5. The teacher might reinforce this point by asking students to build the package that they think is best and consider, for instance, the amount of additional material, or overlap, needed to hold the package together. Does, for example, a $3 \times 3 \times 2$ eighteen-pack need more overlap than a $3 \times 6 \times 1$ eighteen-pack? Or does an eighteen-pack require thicker packaging material than the standard twelve-pack?

This sheet presents opportunities for challenging better students. Students can algebraically analyze the formula that describes the efficiency of a rectangular package when the criterion is minimizing the package material used per can. For example, consider a rectangular package that holds eighteen cans, with m cans along one side. The surface area of each of the two long sides is $2rhm = (2m)(3.2)(12)$. If m cans are along one side, then $18/m$ cans are along the other side. The surface area of each of the two shorter sides is

$$2rh\frac{18}{m} = \left(2\frac{18}{m}\right)(3.2)(12).$$

The surface area of the top, as well as the bottom, is

$$\begin{aligned} lw &= (2rm)\left(2r\frac{18}{m}\right) \\ &= (2m)\left(2\frac{18}{m}\right)(3.2)(3.2). \end{aligned}$$

The total surface area of the rectangular box is then

$$2[(2m)(3.2)(12)] + 2\left[\left(2\frac{18}{m}\right)(3.2)(12)\right] + 2\left[(2m)\left(2\frac{18}{m}\right)(3.2)(3.2)\right].$$

The efficiency is the surface area divided by 18, the number of cans in the package:

$$e = \frac{2((2m)\cdot 3.2 \cdot 12)}{18} + \frac{2\left(\left(2\frac{18}{m}\right)\cdot 3.2\cdot 12\right) + 2\left((2m)\left(2\frac{18}{m}\right)\cdot 3.2\cdot 3.2\right)}{18}.$$

A little simplification produces the equivalent formula

$$e = \frac{25.6m + \frac{460.8}{m} + 245.76}{3}.$$

A graph of this function, shown in **figure 3,** is interesting. Tracing the graph shows that the minimum amount of package material per can occurs when $m \approx 4.24$, which approximates the exact minimum $m = \sqrt{18}$. In other words, students who have concluded that the least packaging material occurs when the rectangular package is as close as possible to a square, in cross section, are indeed correct.

When they complete **sheet 2,** students have had only partial success with the program of minimization and maximization. They have successfully identified the number of cans as an important factor when the criterion is minimizing the package material. But they have seen that changing the number of cans does not help improve efficiency when the criterion is maximizing the package space used by the cans, within the restriction of using rectangular packaging.

Sheet 3

Since associating the number of cans in the package with packaging material used per can has proved fruitful, **sheet 3** is concerned only with maximizing the package space used by the cans. Before beginning this activity sheet, students should understand that their mathematical conclusions have shown that the identification of the number of cans as a controlling factor, when the criterion is maximizing the package space, is incorrect. Rejecting modeling results means that backtracking is necessary; other factors must be considered in this situation. The purpose of this sheet is to consider package shape as a controlling factor.

Although this sheet does not ask questions about minimizing packaging material, you may want to ask them of your students. If your students have

Mathematical results need interpretation and evaluation

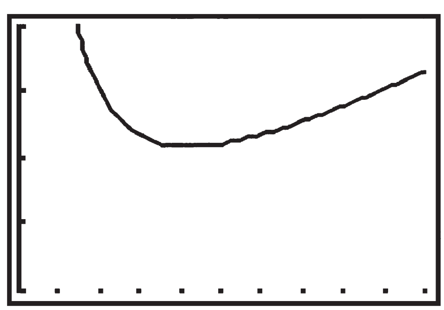


Fig. 3
$[0, 10] \times [100, 200]$

not worked with the surface area of a cylinder, finding the amount of packaging material needed for the package in question 1 prepares them for the assessment problem.

Although the answers given for questions 1 and 2 are analytical, the questions themselves do not demand this type of approach. Less sophisticated students can construct the figures, perhaps using an electronic drawing utility, and calculate efficiencies from measurements. This approach is realistic from a modeling perspective; modelers often apply technology when mathematical analysis proves too difficult.

Question 4 of this sheet is an open-ended miniproject and is an excellent small-group activity. For ease of experimentation, consider having students use the previously described circular manipulatives. They can create posters or transparencies to share their designs and defend their calculations. Also consider asking students to confirm their theoretical calculations by using an electronic drawing utility to make accurate constructions.

Assessment

Because question 4 of sheet 3 is a miniproject, verbal or written reports discussing question 4 may furnish sufficient assessment for this activity. As an additional assessment, the teacher can show the photograph in **figure 4** or describe it to students and ask them to evaluate the two designs. Both coolers are designed to hold six soft-drink cans. The problem might be titled "The Tube or the Cube?" although students should understand that the rectangular design is not cubical.

A model answer to this assessment should indicate that when the criterion is the percent of package space used, the tube is perfect, since the cans use 100 percent of the available space. Students are likely to recall, without calculation, that the efficiency of the "cube" is about 78.5 percent.

A model answer should consider more than one criterion. Students may need to be reminded that a standard can has a radius of 3.2 cm and a height of 12 cm. Students should use those dimensions to calculate that the "cube" uses approximately

$$\frac{2(4 \cdot 3.2 \cdot 12) + 2(6 \cdot 3.2 \cdot 12) + 2(4 \cdot 3.2 \cdot 6 \cdot 3.2)}{6},$$

or 209.92, square centimeters of packaging material per can. The tube, which is a cylinder with a radius of 3.2 cm and a height of 72 cm, uses approximately

$$\frac{2\pi(3.2^2) + 72(\pi \cdot 6.4)}{6},$$

or 252, square centimeters of packaging material per can.

If you have emphasized that mathematical results need to be interpreted and evaluated in the real world, an exemplary answer might address other concerns. For example, since the tube has greater surface area than the "cube," it does not insulate as well. The tube may be inconvenient for such other reasons as its comparative inability to remain stationary.

Modelers often apply technology when mathematical analysis proves too difficult

Fig. 4
The tube or the "cube"?

SOLUTIONS

Sheet 1

1. *a*) 62.6 billion ÷ 12 ≈ 5.22 billion twelve-packs; 5.22 billion × \$.001 ≈ \$5.22 million.
   b) A little over a half-million dollars (\$522 000)
2. *a*) One example is the percent of package space used by the cans; another appropriate measure would be cubic units per can.
   b) The space used might be measured in square inches per package if the packages all hold the same number of cans. Otherwise, it could be measured in square inches per can.
   c) Attractiveness is subjective and is therefore difficult to measure. Perhaps a statistical experiment could be designed in which a random sample of consumers is asked to pick the most appealing of several designs.
   d) The material used could be measured in cubic inches. If the thickness of the material is constant, square inches could be the measure.

Measuring in cubic or square units per can is a better choice, since it allows comparing designs for different numbers of cans.

e) The cost could be measured in dollars or cents per can.

3. *a*) $$\frac{12(3.2^2\pi)}{(6 \cdot 3.2)(8 \cdot 3.2)} \approx 0.785$$

The cans use about 78.5 percent of the package space.

b) $$\begin{aligned}\frac{12(r^2\pi)}{(6 \cdot r)(8 \cdot r)} &= \frac{12\pi r^2}{48r^2} \\ &= \frac{\pi}{4} \\ &\approx 0.785\end{aligned}$$

c) $$\begin{aligned}\frac{12\pi r^2 h}{(6 \cdot r)(8 \cdot r)h} &= \frac{12\pi r^2 h}{48r^2 h} \\ &= \frac{\pi}{4} \\ &\approx 0.785,\end{aligned}$$

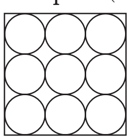
or about 78.5 percent.

4. *a*) $2(6 \cdot 3.2 \cdot 12) + 2(8 \cdot 3.2 \cdot 12) + 2(6 \cdot 3.2 \cdot 8 \cdot 3.2) = 2058.24 \text{ cm}^2$.

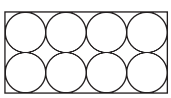
b) For comparison purposes, material per can is a better measure. The standard twelve-pack uses approximately 171.5 square centimeters per can; the competing design uses approximately 187.5 square centimeters per can. The standard twelve-pack is more efficient.

Sheet 2

1. Sample answer: a nine-pack (3 cans × 3 cans)



Second sample answer: an eight-pack (2 cans × 4 cans)



2. For the nine-pack:

$$\begin{aligned}\frac{9\pi r^2}{(6r)(6r)} &= \frac{9\pi r^2}{36r^2} \\ &= \frac{\pi}{4} \\ &\approx 0.785\end{aligned}$$

For the eight-pack:

$$\begin{aligned}\frac{8(\pi r^2)}{(4r)(8r)} &= \frac{8\pi r^2}{32r^2} \\ &= \frac{\pi}{4} \\ &\approx 0.785\end{aligned}$$

The number of cans is apparently not a controlling factor if the criterion is maximizing the package space used by the cans.

3. For the nine-pack, $2(6 \cdot 3.2 \cdot 12) + 2(6 \cdot 3.2 \cdot 12) + 2(6 \cdot 3.2 \cdot 6 \cdot 3.2)$, or 1658.88, square centimeters; per can, 184.32 square centimeters

For the eight-pack, $2(4 \cdot 3.2 \cdot 12) + 2(8 \cdot 3.2 \cdot 12) + 2(4 \cdot 3.2 \cdot 8 \cdot 3.2)$, or 1576.96, square centimeters; per can, 197.12 square centimeters

The amount of packaging material used per can varies, so the number of cans is a controlling factor when the modeling criterion is minimizing the packaging material used per can.

4. In general, increasing the number of cans decreases the amount of packaging material used per can. However, for two packages with the same number of cans, a configuration in which the ratio of length to width is closest to 1 is better. This ratio is sometimes called the *aspect ratio*. For example, a 4 × 4 sixteen-pack uses 158 square centimeters per can; a 2 × 8 sixteen-pack uses 177.92 square centimeters per can.

5. According to the modeling results, a 6 × 3 package will use less packaging material per can than a 2 × 9 package. Calculations can confirm this result.

For a 6 × 3 package:

$$\frac{2(12 \cdot 3.2 \cdot 12) + 2(6 \cdot 3.2 \cdot 12)}{18} + \frac{2(12 \cdot 3.2 \cdot 6 \cdot 3.2)}{18} = 158.72 \text{ cm}^2 \text{ per can}$$

For a 2 × 9 package:

$$\frac{2(4 \cdot 3.2 \cdot 12) + 2(18 \cdot 3.2 \cdot 12)}{18} + \frac{2(4 \cdot 3.2 \cdot 18 \cdot 3.2)}{18} = 175.79 \text{ cm}^2 \text{ per can}$$

However, a more insightful student should realize that stacking the cans in two 3 × 3 layers produces even better results:

$$\frac{2(6 \cdot 3.2 \cdot 24) + 2(6 \cdot 3.2 \cdot 24)}{18} + \frac{2(6 \cdot 3.2 \cdot 6 \cdot 3.2)}{18} = 143.36 \text{ cm}^2$$

When the modeling criterion is maximizing the package space used by the cans, stacking the cans in two layers has no effect: the cans still use 78.5 percent of the available space.

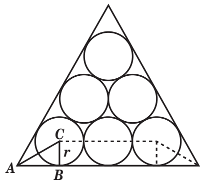
Sheet 3

1. $$\begin{aligned}\frac{7\pi r^2}{\pi(3r)^2} &= \frac{7\pi r^2}{9\pi r^2} \\ &= \frac{7}{9} \\ &\approx 0.778,\end{aligned}$$

or about 77.8 percent. It is slightly less efficient than the standard twelve-pack.

➔

2.



In the figure, $CA = 2r$; $AB = r\sqrt{3}$. The base of the triangular package is $4r + 2r\sqrt{3}$. Since the triangle is equilateral, its area is

$$\frac{s^2\sqrt{3}}{4} = \frac{(4r + 2r\sqrt{3})^2\sqrt{3}}{4}.$$

The percent of space used by the cans is

$$\frac{6\pi r^2}{\frac{(4r + 2r\sqrt{3})^2\sqrt{3}}{4}} = \frac{6\pi r^2}{1} \bullet \frac{4}{4r^2(2+\sqrt{3})^2\sqrt{3}}$$

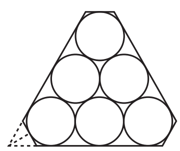
$$= \frac{6\pi}{(2+\sqrt{3}^2\sqrt{3}}$$

$$\approx 0.781,$$

or about 78.1 percent. This design is slightly less efficient than the standard twelve-pack, but it is slightly more efficient than the cylindrical seven-pack.

Students should be skeptical about generalizing this principle to packaging that is based on some other shape

3. On the basis of their experiences with rectangular packaging, students will probably expect that the number of cans is not a factor when the criterion is the percent of package space used by the cans. Although this expectation is a reasonable one, you can encourage students to be skeptical about whether this principle generalizes to packaging that is based on a triangle or on some other shape. Students should also realize that not every number of cans can be enclosed in an equilateral triangle.
4. Sample answer: Remove three small equilateral triangles from the triangular six-pack to form an irregular hexagon.



Each small equilateral triangle (one of which is shown at the lower left in the figure) has an altitude of length r. Therefore, the length of each side of the triangle is $2r/\sqrt{3}$. The area of each small triangle is

$$\left(\frac{2r}{\sqrt{3}}\right)^2\left(\frac{\sqrt{3}}{4}\right) = \frac{r^2\sqrt{3}}{3}.$$

The area of the irregular hexagon is

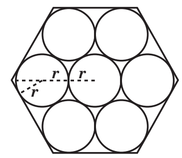
$$\frac{(4r + 2r\sqrt{3})^2\sqrt{3}}{4} - 3\left(\frac{r^2\sqrt{3}}{3}\right) = r^2((2+\sqrt{3})^2\sqrt{3} - \sqrt{3}).$$

The efficiency of the package is

$$\frac{6\pi r^2}{r^2((2+\sqrt{3})^2\sqrt{3} - \sqrt{3})} = \frac{6\pi}{(2+\sqrt{3})^2\sqrt{3} - \sqrt{3}}$$

$$\approx 0.842.$$

Second sample answer: A hexagonal seven-pack



The hexagon can be divided into six equilateral triangles, each with sides measuring

$$2r + \frac{2r}{\sqrt{3}}.$$

Therefore, the area of the hexagon is

$$6\left[\frac{\left(2r + \frac{2r}{\sqrt{3}}\right)^2\sqrt{3}}{4}\right] = \frac{\left(2r + \frac{2r}{\sqrt{3}}\right)^2 3\sqrt{3}}{2}.$$

The package space used by the cans is

$$\frac{7\pi r^2}{\frac{\left(2r + \frac{2r}{\sqrt{3}}\right)^2 3\sqrt{3}}{2}} = \frac{14\pi}{3\sqrt{3}\left(2 + \frac{2}{\sqrt{3}}\right)^2}$$

$$\approx 0.851,$$

or about 85.1 percent.

BIBLIOGRAPHY

Carter, Claudia R. "Activities: Packing Them In." *Mathematics Teacher* 90 (March 1997): 211–14, 220–22.

Consortium for Mathematics and Its Applications (COMAP). "The Right Stuff." In *Mathematics: Modeling Our World: Course 2*, pp. 254–319. Cincinnati, Ohio: South-Western Educational Publishing, 1998.

Giordano, Frank R., Maurice D. Weir, and William P. Fox. *A First Course in Mathematical Modeling*. 2nd ed. Pacific Grove, Calif.: Brooks/Cole Publishing Co., 1997.

Hine, Thomas. *The Total Package*. Boston: Little, Brown & Co., 1995.

In this activity, you will use mathematical modeling to describe and improve the efficiency of secondary soft-drink packaging. The primary packaging is the can; the secondary packaging is the container that holds several cans.

Before beginning the modeling process, consider the impact that an improvement in the design of soft-drink packages might have. Because the volume of soft drinks consumed is large, even a small savings on the cost of a secondary package could mean a lot to the soft-drink industry.

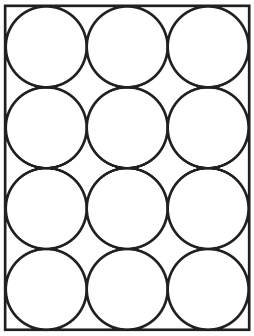
1. According to the National Soft Drink Association, 62.6 billion cans of soft drinks were consumed in the United States in 1995. Suppose that you find a way to save the soft-drink industry one-tenth of a cent ($.001) on the packaging of each twelve-pack sold.
   a) Estimate the total annual savings to the soft-drink industry.
   ____
   b) If you receive royalties worth 10 percent of the savings to the industry, estimate your annual income for the use of your innovation in the United States.
   ____
2. The first step in the mathematical-modeling process is to identify a real-world problem and define the modeling goal clearly. This example considers the efficiency of secondary packaging of soft drinks, and the modeler must decide how to measure efficiency. For example, if the primary concern of someone who is trying to improve the efficiency of automobile engines is economy, efficiency might be defined in terms of gasoline consumption and measured in miles per gallon. However, if the primary concern is the environmental impact of automobiles, efficiency might be defined in terms of emissions and measured in parts per million.

   Modeling often involves optimization—finding the best way to do something. For each of the following criteria for optimal packaging, give at least one example of how efficiency could be measured.
   a) The best soft-drink package is one in which the cans use as much of the available space in the package as possible. Hint: What numerical measure would indicate how well the cans use the space in the package?
   b) The best soft-drink package is one that uses shelf space well.
   c) The best soft-drink package is one that is most attractive to consumers.
   d) The best soft-drink package is one that uses the least packaging material.
   e) The best soft-drink package is one for which the packaging material is cheapest.

From the *Mathematics Teacher,* September 2000

Modelers use various mathematical tools to reach conclusions. Questions 3 and 4 show how geometric formulas can help calculate packaging efficiency. Questions 3 and 4 also introduce simplification, a necessary part of the modeling process.

3. Mathematical modeling always involves simplification. Simplification is often done to eliminate relatively unimportant information, but it is also done for convenience. The figure shown is a two-dimensional representation of the twelve-pack commonly used for secondary packaging of soft drinks.



a) The radius of a soft-drink can is approximately 3.2 centimeters. Determine how well the cans use the space in this two-dimensional package, that is, calculate the percent of package space used by the cans.

b) Show that your answer is independent of the size of the cans, that is, repeat your work, but use r to represent the radius.

c) Show that simplification to two dimensions does not affect the result, that is, find the volume of the package and the volume of the cans in terms of the radius r and height h of the cans, then calculate the percent of space used by the cans.

4. *a*) Determine the amount of packaging material used by the standard twelve-pack, as illustrated in the figure. For simplicity, assume that the material has no thickness and use the area of the packaging material as your measure. Also assume that the package requires no overlap of packaging material. Note that the height of a soft-drink can is about 12 centimeters.

b) Suppose that a competing design uses 1500 square centimeters of packaging material to hold eight cans. Is this design more efficient than the standard twelve-pack? Explain.

From the *Mathematics Teacher,* September 2000

You have assessed the efficiency of a standard twelve-pack by two measures: the percent of package space used by the cans and the packaging material used per can. In the first case, the modeling objective is maximization; in the second case, the modeling objective is minimization.

After the goal of the modeling process has been defined, the modeler must identify important factors that control the chosen criterion, then conduct a mathematical investigation. For example, a modeler might identify temperature as a factor that controls the chirp rate of crickets and choose to ignore other factors until the effect of temperature is understood. If accurate predictions of the chirp rate of crickets can be made from temperature alone, no other factors need to be considered.

Your task here is to consider the number of cans in the container as a factor that controls a criterion for optimal packaging. To assess the effect of this factor, no others should be considered, that is, only the number of cans will vary.

1. Design a secondary package in which the number of cans is not twelve. Remember, everything else should remain the same. Do not change the size or the shape of the can. Do not change the shape of the package—keep it rectangular.

2. Find the percent of packaging space used by the cans in your design. Compare the percent with that used by other designs in your class. Is the number of cans a controlling factor if the criterion is maximizing the percent of packaging space used by the cans?

3. Find the amount of packaging material used per can in your design. Compare the amount with the amounts used in other designs that your class has created. Is the number of cans a controlling factor if the criterion is minimizing the amount of packaging material used per can?

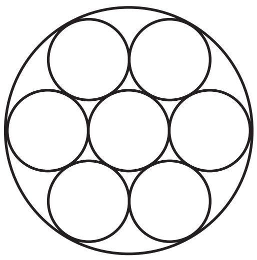
4. Describe the effect of the number of cans as a controlling factor. Interpret the mathematical results that your class has produced, and recommend a package design.

5. Results produced by the mathematical-modeling process must be tested against reality. In the example of soft-drink packaging, the model may not address such considerations as convenience. For example, if you concluded that a package should contain a relatively large number of cans, some consumers may find the package too heavy or think that it requires too much storage space. Use your modeling results to develop a recommendation for a soft-drink company that has decided that a rectangular eighteen-pack is the most convenient.

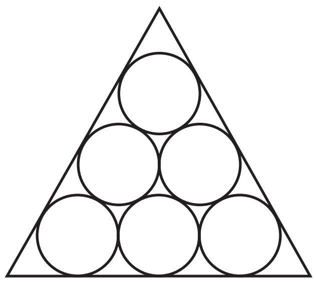
From the *Mathematics Teacher,* September 2000

Your modeling efforts have been successful when the efficiency criterion is minimizing the packaging material used per can. But you have been unsuccessful in improving efficiency when the criterion is maximizing the package space used by the cans. When the mathematical-modeling process fails to produce results, part of the process must be repeated. You next examine another potential controlling factor: package shape.

1. Shown here is a cylindrical seven-pack. Determine the percent of package space used by the cans. How does this design compare with the standard twelve-pack?



2. Shown here is a triangular six-pack. Determine the percent of package space used by the cans. How does this design compare with the standard twelve-pack?



3. Would you expect a triangular three-pack or a triangular ten-pack to make better use of package space than the triangular six-pack? Explain.

4. Design your own package. Try to find a shape for which the percent of space used by the cans is greater than that of any of the designs that you have evaluated.