

1 Developing Algebraic Habits of Mind

A Framework for Classroom Questions Aimed at Understanding Student Thinking

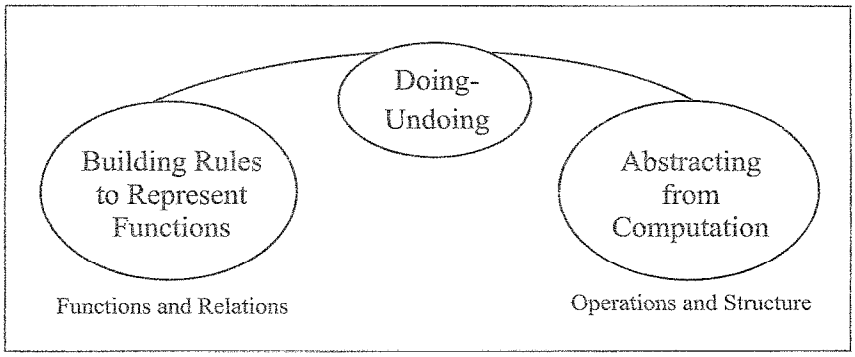
Because algebra comprises so many mathematical features, the term *algebraic thinking* defies simple definition. Generally, those who use the term do so after first choosing to concentrate on particular features, and then concern themselves with the thinking that those features demand. For example, some focus on the abstract features that distinguish algebra from arithmetic. With that perspective, they might characterize algebraic thinking as "the ability to operate on an unknown quantity as if the quantity was known, in contrast to arithmetic reasoning which involves operations on known quantities" (Langrall & Swafford 1997, 2). Others focus on the important role that functions play in algebra, and may characterize algebraic thinking as the capacity to represent quantitative situations so that relations among variables become apparent. Yet others may have problem solving as their point of reference for thinking about algebra and for thinking algebraically, and might concern themselves with how problem solvers model problems.

Our perspective has been influenced by our work with groups of teachers representing grades 6 through 10, so we emphasize habits of thinking that can begin developing in the prealgebra years and, if nurtured, can serve the learning of formal algebra as well. When people think algebraically in order to solve problems, explore, and so on, certain habits of thinking come into play. This chapter discusses three habits that seem to be critical to developing power in algebraic thinking. The list isn't meant to be comprehensive. However, we have no doubt that by learning to attend, in an ongoing fashion, to these several habits—in our own and in students' mathematical work—we will be better prepared to help students succeed in algebra.

A facility with algebraic thinking includes being able to think about *functions* and how they work, and to think about the impact that a system's *structure* has on calculations. These two aspects of algebraic thinking are facilitated by certain habits of mind (Figure 1–1):

- **Doing–Undoing.** Effective algebraic thinking sometimes involves reversibility (i.e., being able to undo mathematical processes as well as do them). In effect, it is the capacity not only to use a process to get to a goal, but

FIGURE 1–1. *Three Alegbraic Thinking Habits of Mind*



also to understand the process well enough to work backward from the answer to the starting point. So, for example, in a traditional algebraic setting, algebraic thinkers cannot only solve an equation such as $9x^2 - 16 = 0$, but also answer the question, "What is an equation with solutions 4/3 and –4/3?"

- **Building Rules to Represent Functions.** Critical to algebraic thinking is the capacity to recognize patterns and organize data to represent situations in which input is related to output by well-defined functional rules. For example, here is a functional rule that is computation-based: "Take an input number, multiply it by 4 and subtract 3." This habit of mind is a natural complement to Doing–Undoing, in that the capacity to understand how a functional rule works in reverse generally makes it a more accessible and useful process.

- **Abstracting from Computation.** This is the capacity to think about computations independently of particular numbers that are used. One of the most evident characteristics of algebra has always been its abstractness. But, just what is being abstracted? To answer this, a good case can be made that thinking algebraically involves being able to *think about computations freed from the particular numbers they are tied to in arithmetic*—that is, abstracting system regularities from computation. For example, students invoke this habit of mind when they realize that they can regroup numbers into pairs that equal 101 to make the following computation simpler: "Compute: 1 + 2 + 3 + . . . + 100." There is a suggestion of Doing–Undoing here, as well, in the recognition that 101 can be decomposed into 100 + 1; 99 + 2; 98 + 3; and so on.

Guiding Questions

Habits of mind develop as the thinker pays attention, over and over again, to "what works" (e.g., what helps in solving problems or what can explain the regularity in a particular pattern) and looks for cues in new situations that previously used ap-

proaches may help. Often, the cueing may occur through "guiding questions" that the thinker asks himself. For example, consider these three basic algebraic-thinking questions: How does this process work in reverse? How are things changing in this situation? What are my operation shortcut options to get from here to there? The first and second questions may spur the representation of functions, when that is appropriate; the third may help to spur abstract thinking about calculation, when that is appropriate.

Table 1–1 is a beginning list of guiding questions for each of the three habits of mind. Throughout this book, we try to illustrate the usefulness of the questions with various mathematics activities and examples of student work. The questions in Table 1–1 have developed in our projects over time, and they keep developing and changing as teachers engage with the notion of algebraic-thinking habits of mind and experiment with questions that can help to foster their development in students.

The Role of Classroom Questions

If, as is our belief, these habits of algebraic thinking can be learned, what should teachers be doing to foster the learning? Based on our best information, we can say that productive instruction probably combines the following:

- Consistent modeling of algebraic thinking. For example, in summarizing student responses to a mathematical activity, a teacher might try to make explicit what students have left implicit in their thinking: "So, you decided to try your rule on some larger numbers to see if it would work."
- Giving well-timed pointers to students that help them shift or expand their thinking, or that help them pay attention to what is important. For example, the students of a Linked Learning middle-grades teacher were working on an activity that could be completed satisfactorily via arithmetic. The teacher saw an opportunity for algebraic thinking and said, "Once you have made a chart, look for an easier way. Pay attention to how the numbers group and how the groupings might suggest an easier way."
- Making it a habit to ask a variety of questions aimed at helping students organize their thinking and respond to algebraic prompts. For example, we have noted among some Linked Learning teachers a consistent use of questions that challenge students to analyze expressions: "Can you explain what the 3 and the 5 represent in that equation?"

We have been paying particular attention to the value of questioning. A couple of beliefs have grown out of the work we have done in our teacher-enhancement projects. One has to do with the role of *intention* in teachers' questioning; the other has to do with the mathematical *context* in which the question is asked:

TABLE 1–1. *Guiding Questions*

Questions for Doing—and Undoing	Questions for Building Rules to Represent Functions	Questions for Abstracting from Computation
How is this number in the sequence related to the one that came before?	Is there a rule or relationship here?	How is this calculating situation like/unlike that one?
What if I start at the end?	How does the rule work, and how is it helpful?	How can I predict what's going to happen without doing all the calculation?
Which process reverses the one I'm using?	Why does the rule work the way it does?	What are my operation shortcut options for getting from here to there?
Can I decompose this number or expression into helpful components?	How are things changing?	When I do the same thing with different numbers, what still holds true? What changes?
	Is there information here that lets me predict what's going to happen?	What are other ways to write that expression that will bring out hidden meaning?
	Does my rule work for all cases?	How can I write the expression in terms of things I care about?
	What steps am I doing over and over?	How does this expression behave like that one?
	Can I write down a mechanical rule that will do this job once and for all?	
	How can I describe the steps without using specific inputs?	
	When I do the same thing with different numbers, what still holds true? What changes?	
	Now that I have an equation, how do the numbers (parameters) in the equation relate to the problem context?	

1. **Intention.** It is valuable for teachers to be aware of the variety and breadth of intention behind classroom questions and to seek, over time, patterns of questioning that are balanced across the range of intention.
2. **Context.** Questions aimed at developing students' algebraic thinking patterns should be asked in situations that are patently "algebraic," as well as in situations in which the relevance of algebraic thinking isn't as obvious.

Ideally, once the teacher is able to concentrate wholly on algebraic thinking, the questions will sound like these adaptations from the questions listed in Table 1–1:

- Which process reverses the one you're using?
- How does the rule work?
- How are things changing?
- Can you find a more helpful way to write the rule?
- Is there information here that lets you predict what's going to happen?
- How can you predict what's going to happen without doing all the calculation?
- What are your operation shortcut options for getting from here to there?
- When you do the same thing with different numbers, what still holds true? What changes?

Of course, teachers in real classrooms are dealing with factors that often make the ideal seem remote and, perhaps, unrealistic. Usually, considerable groundwork must be laid for asking algebraic-thinking questions.

Intention

To get a handle on the kinds of questions that lay this groundwork, the Linked Learning Project relied on a cadre of teacher leaders who act as classroom observers of the project's teachers, and asked them to pay special heed to the questions asked by the teachers and the impact of these questions on students. The observations take place in classes in which the teachers are using a lesson meant to elicit algebraic thinking from the students.[1] In post-observation debriefings, the observer checks with the teacher on the accuracy of the observer's judgment about the intended purpose of each recorded question. From the observation data, we found that teachers' questions fall into five categories, according to a teacher's general intention in asking them (Table 1–2).

Any effective lesson or set of lessons will use a blend of question types. Because students can need clarification on what a mathematical challenge is asking them to do, because their attention may need orienting toward key features, and because their underlying reasoning may not be clear or may seem incomplete or faulty, teachers need to use a variety of questions or prompts that are not particularly algebraic, but that lay groundwork for and, ideally, foreshadow students' algebraic thinking.

Context

In addition to teacher intention, mathematical context is another consideration in teachers' use of classroom questions to elicit algebraic thinking and, over time, to foster the development of algebraic habits of mind. Some of the activities that teachers use will display their algebraic potential rather explicitly. For example, there is a clear call for finding a rule to describe a pattern ("What is a general way

TABLE 1–2. *Five Categories of Teacher's Questions*

Question Type	Examples
Managing Intended to help set students on task, get their work organized, etc.	Who's in charge of writing it down? Are you guys working? What are you doing now?
Clarifying Intended to request information from the student when the teacher isn't clear about what the student means or intends; also, when the teacher is trying to help the student clarify the question	Do you know what *perimeter* is? How did you get 2? (This is asked when the teacher is trying to follow the student's thinking, not trying to correct thinking or help refocus it in a different direction.) Who went first? (during a mathematics game)
Orienting Intended to get students started, or to keep them thinking about the particular problem they are solving; may suggest ways to focus on the problem; also, to orient and/or motivate the student toward the correct answer or away from the incorrect answer	What's the problem asking you to find? Have you thought about trying a table? If you have that number and it increases 3, what do you get? (emphasis on error) How did you get 18 (when the answer is some other value)? How can you check your answer? (wrong answer)
Prompting Mathematical Reflection Intended to ask students to reflect on and explain their thinking; to have them understand others' mathematical ways of thinking; and to have them extend their thinking about the mathematics in a problem	How do you explain that? Can you explain how you got the values in the table? Why did the two of you reach different conclusions? Can you estimate? . . . Now check. Does anyone have a different way?
Eliciting Algebraic Thinking Intended to ask the students to undo, to build rules for describing functional relationships; to abstract from computations they have made; to ask about the meaning of the work they're doing; to ask about what statements are "always" true, about *n*th terms, and about finding patterns and looking for what changes; to work forward and backward, etc.; and to ask students to justify generalizations	What could it (the value in the equation) represent? How could you use the formula? In x years, how much does it go up? Can you look for a pattern? Find out how the rule works. What does –2 mean? If this is 13 and this is 16, by how much did it increase? (emphasis on change) What is an easier way? Pay attention to how the numbers group.

to say how high the tree will be after n months?"), or students are asked to make a general statement drawn from a particular calculation (e.g., "$5 = 9 - 4$. How many odd numbers can be written as the difference of two perfect squares? Show why you think you have them all."). In such cases, teacher questions can reinforce students' appreciation of what are important features, such as

- comparing the relative value of different representations of a relation: "What does the graph tell you? Now, what different information does the equation give you?"
- looking back, after a solution, for a shortcut that may have been missed and could be useful next time: "How could you have reached that conclusion without looking at the chart?"
- making sure that all the relevant cases have been found: "How can you be sure you have found all the numbers that work?"
- seeing what happens when other numbers are tried: "What if you try a much larger number there?"

In other activities, it is likely that the full algebraic potential—or, in many cases, *any* algebraic potential—will go unexploited unless the teacher asks questions that are used to extend students' thinking about the problem. In those cases, the teacher may be

1. reversing a routine calculating task to challenge students to undo as well as do: "Now that you have a good handle on using a factor tree, answer this: What whole numbers have three factors, including the factor of 1?"
2. asking "what if" questions to extend beyond a single situation to a more generalized situation. For example, suppose students have solved the following problem:

Golden Apples[2]

A prince picked a basketful of golden apples in the enchanted orchard. On his way home, he was stopped by a troll who guarded the orchard. The troll demanded payment of one-half of the apples plus two more. The prince gave him the apples and set off again. A little further on, he was stopped by a second troll guard. This troll demanded payment of one-half of the apples the prince now had plus two more. The prince paid him and set off again. Just before leaving the enchanted orchard, a third troll stopped him and demanded one-half of his remaining apples plus two more. The prince paid him and sadly went home. He had only two golden apples left. How many apples had he picked?

Rather than settle for the solution only, the teacher can further students' thinking by asking, "What if he had 4 left? How many did he begin with? 6 left?", and so on.

3. exploiting calculating situations in which there is regularity, to challenge students to use calculating shortcuts based on the regularity (e.g., "Without writing out all the numbers and adding them, find the total: $1 + 2 + 3 + \ldots + 27 + 28 + 27 + \ldots + 3 + 2 + 1$.")

4. exploiting calculating situations in which there is regularity, to challenge students to make general statements (e.g., "Think of three consecutive integers and multiply them. Does 2 divide any such product? Why? What other integers divide any such product? What is the largest integer that you can be certain divides any such product evenly? Why?")

Analyses of student work can support the use of classroom questions to foster algebraic thinking, in particular, prompting reflections about appropriate questions. For example, consider Figure 1–2, a piece of student work, drawn from a Linked Learning classroom.

After examining the sample in Figure 1–2, ask yourself what you notice and what it makes you wonder about. What do you infer about the student's line of thinking? What is noticed, inferred, or wondered about can lead to productive instructional questions. What questions might you ask in order to help the student push her thinking further?

One feasible inference is that the student is onto something productive in the last answer, something like, "When I divide the number of the square into the number of its toothpicks, I get 4, 6, 8, 10 for the four squares that I have." From a habit-of-mind perspective, it seems that the student has wondered, "How are things changing?"—a key question in Building Rules to Represent Functions. It also seems that she has done some fluid Doing–Undoing to test how 4, 12, 24, and 40 could be generated from, respectively, 1, 2, 3, 4.

The following are questions that might push the student further: "What have you done to get these numbers?" "When you were counting toothpicks, were you using any counting shortcuts?" "What information is here to help you predict what's going to happen in the next squares?" In "later squares?"

Example Activities

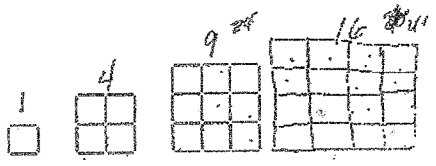
In this section, we offer several mathematical activities, which we believe can illustrate some of our points about algebraic thinking and the role of classroom questioning. Furthermore, on the basis of the experience of teachers in our projects, we are confident that these activities can be used with students in grades 6 through 10.

FIGURE 1–2. *Student Work from a Linked Learning Classroom*

Toothpick Squares Name_____

Shown below is a pattern of "growing" squares made from toothpicks.

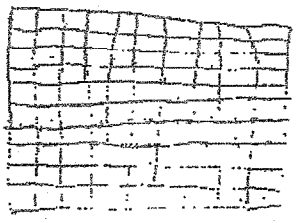
1. Study the pattern and draw a picture of the next likely shape in the pattern.



2. How many small squares make up the new square?

16 Small squares

3. How many small squares would make up a large square which has 10 toothpicks on each side? (Show your work.) 100 square



4. Write a rule which will allow you to find the number of small squares in any large square.

1-1
2-4
3-9
4-16
5-25
6-36
7-49
8-64
9-81
10-100

they have to time the number by 1 sets

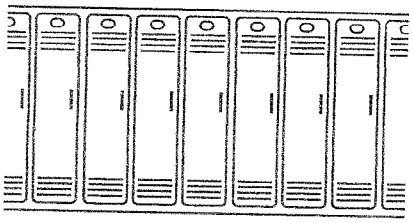
5. Find a rule which will let you find the number of toothpicks in any large square. (Show your work.)

1-4(4)
2-12(6)
3-24(8)
4-40(10)

The Locker Problem

The potential for algebraic thinking in this problem is substantial but somewhat hidden. At first, and even second, glance, the problem may seem no more than a number-sense problem, involving knowledge about factoring whole numbers.

The Locker Problem[3]



There are 20 lockers in one hallway of the King School. In preparation for the beginning of school, the janitor closed all of the lockers and put a new coat of paint on the doors, which are numbered from 1 to 20.

When the 20 students from Mrs. Mahoney's class returned from summer vacation, they decided to celebrate by working off some energy. They came up with a plan: The first student ran down the row of lockers and opened every door. The second student started with locker #2 and closed every second door. The third student started with locker #3 and changed the state of every third locker door. The fourth student started with locker #4 and changed the state of every fourth locker door, the fifth student started with locker #5 and changed the state of every fifth locker door, and so on, until all 20 students had passed by the lockers.

Which lockers are still open after the twentieth student is finished? Which locker or lockers changed the most?

Suppose there are 200 lockers. Which lockers are open after the 200th student is finished? Which locker or lockers changed the most?

The number of times a locker changes state is the same as the number of factors of the number on the locker: For example, locker number 12 has factors 1, 2, 3, 4, 6, and 12, so it changes state six times. A locker whose number has the most factors is the locker that changes the most. For the 20-locker case, lockers numbered 12, 18, and 20 share this honor. Some students think, by extension, that, for 200 lockers, lockers numbered 120, 180, and 200 change the most. Why would they think this? Why might they be wrong? This starts to get at why we like the potential for algebraic thinking here, because it raises questions about the function whose input is a whole number N and whose output is the number of factors of N. This is usually called the nu function, after the Greek letter ν. Using this terminology, $\nu(12) = 6$. If you sketch the graph of ν, it moves up and down rather wildly. Consequently, you would not expect ν to be describable by any suc-

cinct rule. However, it turns out that any whole number N can be factored into its "prime factorization," $p_1^{k1}, \ldots, p_m^{km}$, where $p_1, \ldots, p_m$ are distinct primes and $k_1, \ldots, k_m$ are whole numbers. It is the case that $\nu(N)$ equals the product of $(k_1 + 1), \ldots, (k_m + 1)$. For example, the prime factorization of 12 is $2^2 3^1$, so the rule says that $\nu(12)$ should be $(2 + 1)(1 + 1) = 6$, which we already know it is. Why does this formula work for ν? Think about how factors of a number are constructed from their prime factors.

A locker is open at the end if it changes state an odd number of times. If you record the numbers between 1 and 200 with an odd number of factors, you get 1, 4, 9, 16, 25, 36, and so on, namely, the perfect squares. Why might this be the case? Thinking about the formula for ν can give you an idea.

By now, it should be evident how The Locker problem can elicit algebraic thinking, in particular, the habits of mind Building Rules to Represent Functions (in particular, working to represent ν) and Doing–Undoing (addressing reverse questions, such as "What numbers have 6 factors? 3 factors? An odd number of factors?").

Relevant guiding questions, from the list offered earlier:

- Can I write down a mechanical rule that will do this job once and for all?
- How does the rule work, and how is it helpful?
- Why does the rule work the way it does?
- What if I start at the end?
- What process reverses the one I am using?

Crossing the River

This problem has proved to be widely accessible to students in grades 6 through 10. Students can represent the situation in verbal, tabular, pictorial, or symbolic (i.e., with equation) forms.

Crossing the River[4]

Eight adults and two children need to cross a river. A small boat is available that can hold one adult or one or two children (i.e., three possibilities: 1 adult in the boat, 1 child in the boat, or 2 children in the boat). Everyone can row the boat. How many one-way trips does it take for all of them to cross the river? Can you describe how to work it out for 2 children and any number of adults? How does your rule work out for 100 adults?

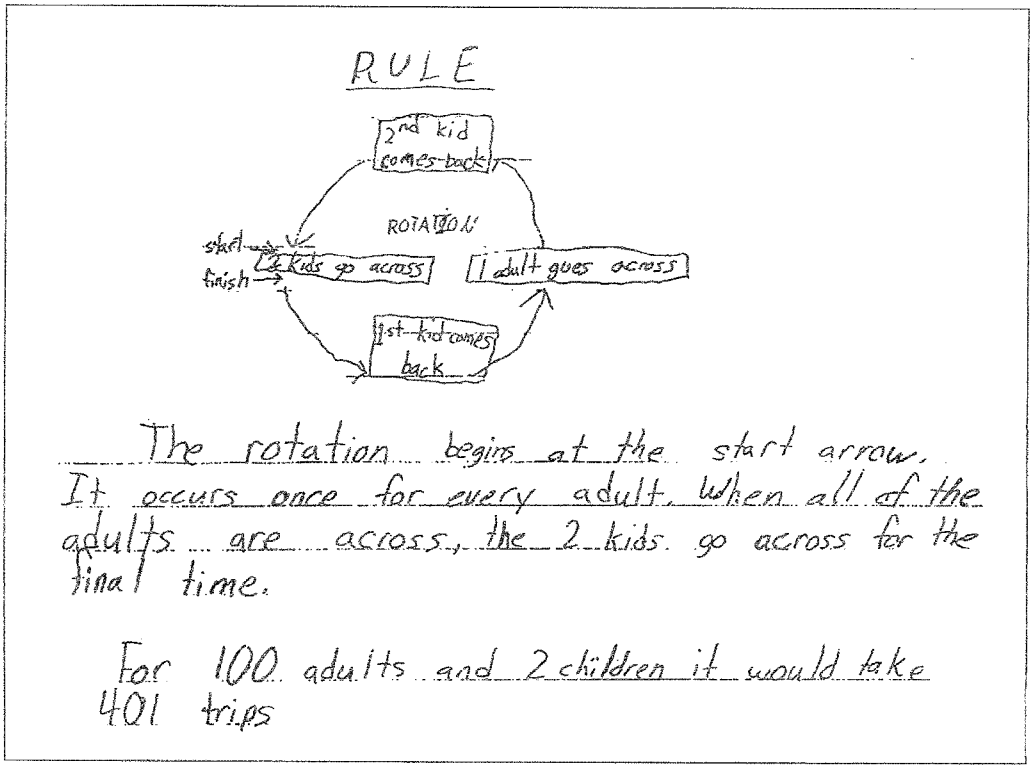
What happens to the rule if there are different numbers of children? For example: 8 adults and 3 children? 8 adults and 4 children? Write a rule for finding the number of trips needed for A adults and C children.

One group of adults and children took 27 trips. How many adults and children were in the group? Is there more than one solution?

Once you see that the same sequence of 4 trips can be repeated over and over again, you can use this chunk to build a rule, and to extend it to any number of adults with 2 children. Students familiar with algebra may symbolize the chunk of 4 in an equation like $4A + 1$ = number of one-way trips. (Question to consider: If the 4 represents the sequence of trips that gets repeated, what does the 1 represent?) However, they may just as easily represent the chunk pictorially, as did the student who drew Figure 1–3.

The introduction of different numbers of children requires a reworking of the rule. This calls into play several of the guiding questions listed earlier. One thing that changes is the number of trips with no adults in the boat. The repeated sequence of 4 trips per adult does not change. A symbolic way to express the rule for A adults and C children: number of one-way trips = $4A + 2C - 3$. (Question to consider: What does $2C - 3$ represent?)

FIGURE 1–3. *Student Pictorial of Rule*



Relevant guiding questions:

- What steps am I doing over and over?
- When I do the same thing with different numbers, what still holds true? What changes?
- Now that I have an equation, how do the numbers (parameters) in the equation relate to the problem context?

Finally, the activity prompts some Doing–Undoing thinking by giving the final state (27 one-way trips) and asking for the starting conditions (i.e., the numbers of children and adults). This tends to be difficult, though not impossible, for middle school students. For more experienced students, it is the kind of situation that shows how powerful algebraic representation can be. For instance, a person who is aware of the $4A + 2C - 3$ expression can use it to work backward from $4A + 2C - 3 = 27$ (or $4A + 2C = 30$). Can you sense, by looking at this, that there is likely to be more than one solution?

Equations such as $4A + 2C = 30$, with integer solutions and integer coefficients, are examples of *Diophantine Equations*, so there are ample extensions to the Crossing the River problem that can spur algebraic thinking.

Cuisenaire Trains

This is an example of an activity that may not look particularly "algebraic" in its context, but it does invite the building of rules to represent functions. It is a rich problem, the solution of which can be obtained by systematically manipulating Cuisenaire rods, recording the results in a two-column table, and looking for patterns.

Cuisenaire Trains[5]

How many Cuisenaire trains can you make, the total length of which is 2? 3? 4? 10? By a "train" of length 3, we mean a row of rods, the length of which is 3, where order counts.

Make a table. Inspection of the table (total length, T, vs. number of trains, N) reveals a doubling pattern that can be expressed symbolically using the equation $N = 2^{T-1}$ (or its recursive version $N_{k+1} = 2N_k$). So, for example, the students can predict that there are $2^{10-1} = 2^9 = 512$ different trains of length 10. One way to justify that there are 2^{T-1} different trains of length T is to think of having a solid rod of length T and a saw to make cuts. For example, for $T = 6$, you have a choice of making a cut, or not, at five different locations; that is, you have 2^5 possible configurations that you can cut with the saw.



This kind of thinking may be beyond students. Instead, they may justify the doubling by examining the process of making trains of length $T + 1$ from the trains of length T by placing a 1-rod in the front of each train and then a 1-rod at the end of each train. You may want to take a moment now to figure out how close this procedure comes to doubling and what modification needs to be made to reveal the doubling.

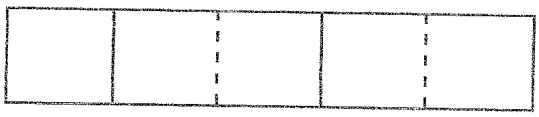
For advanced students, further analysis can reveal there are

$$\binom{T-1}{M-1}$$

trains of length T that are composed of M rods. For example, there are

$$\binom{5-1}{3-1}=\binom{4}{2}=6$$

different trains of length 5 that are composed of 3 rods. They are 122, 212, 221, 113, 131, and 311. Students can justify this result by noting that each of the 6 trains can be thought of as the result of dividing a 5-rod at two of the possible 4-unit lengths. For example, the following diagram shows that the train 122 can be obtained by dividing a 5-rod at the two places indicated by solid lines.



Relevant guiding questions:

- Is there a rule or relationship here?
- How are things changing?
- Is there information here that lets me predict what's going to happen?
- How is this number in the sequence related to the one that came before?

Conclusion

In this chapter and, more broadly, in this book, we advocate for the value of classroom questions in fostering algebraic habits of mind among students in the prealgebra years and beyond. Our framework for shifting classroom practice in this direction isn't based on any attempt to be comprehensive in defining *algebra* or *algebraic thinking*. Rather, it is based on our work with what we believe are three key algebraic habits of mind: Doing–Undoing, Building Rules to Represent Functions, and Abstracting from Computation.

In the following chapters, we address some key issues that arise for teachers in grades 6 through 10 as they try to foster their students' algebraic thinking, in particular, through their classroom questions. The three habits of mind play a key role in our discussions, so we end this chapter with a table for each habit. The information in each table, while not intended to be comprehensive, should provide some concrete examples and connections. Recall what we mean by each of the habits of mind:

- **Doing–Undoing.** Effective algebraic thinking sometimes involves reversibility (i.e., being able to undo mathematical processes as well as do them). In effect, it is the capacity not only to use a process to get to a goal, but also to understand the process well enough to work backward from the answer to the starting point (Table 1–3).
- **Building Rules to Represent Functions.** Critical to algebraic thinking is the capacity to recognize patterns and organize data to represent situations in which input is related to output by well-defined functional rules (Table 1–4).
- **Abstracting from Computation.** This is the capacity to think about computations independently of particular numbers used. One of the most evident characteristics of algebra has always been its abstractness. But, just what is being abstracted? In answer, a good case can be made that thinking algebraically involves being able to think about computations freed from the particular numbers to which they are tied in arithmetic (i.e., abstracting system regularities from computation) (Table 1–5).

Notes

1. Half of the observed lessons in any one teacher's classroom are chosen by the teacher; the other half are activities that are given to the teacher by project staff.
2. From *Make It Simpler*, by Carol Meyer and Tom Sallee; copyright 1983 by Addison–Wesley Publishing Company. Reprinted by permission.
3. This version of the activity was developed by EDC staff for professional development projects. Other versions of the activity can be found in various materials, such as the Grade 6 unit Prime Time, from *Connected Mathematics*. 1998. Menlo Park, CA: Dale Seymour Publications.
4. A similar version of this problem appears in *MathScape: Seeing and Thinking Mathematically, Patterns in Numbers and Shapes, Lesson 3*. 1998. Mountain View, CA: Creative Publications.
5. This version of this problem was developed by Al Cuoco and the EDC staff for EDC professional development projects.

(text continues on page 19)

TABLE 1–3. *Doing–Undoing*

Some Related Mathematical Ideas	Some Possible Indicators	Examples
Inverse Operations; Roots; Functions; Equivalent Expressions		
This habit of thinking need not wait until the topics of functions or solving equations arise. Throughout the elementary and middle school experiences, there are many topics that provide opportunities to foster a Doing–Undoing habit of mind.	Works to relate one entry to previous entries ("How is this number related to the ones that came before?")	How many even numbers are there in the 100th row of Pascal's Triangle? ("You can build the 100th row from the 99th row—each entry is the sum of the two entries right above it. . . .")
For example, dividing one number into another can result in a remainder. Working backward from remainders can foster student thinking about the process or algorithm used in the dividing.	Deduces input from given output, or estimates input from output ("What if I start at the end?")	How would you describe all the numbers with exactly three factors? ("Well, two of the factors are 1 and the number itself, so what could the third one look like?") In the Locker problem, which locker(s) do you think change states the most? ("I think that the biggest power of 2 less than 200 may be the answer, because it would have a lot of factors. . . .")
	Uses an inverse process (works backward through a sequence of steps) ("What process reverses the one I'm using?")	When my age is divided by 3, the remainder is 1. When my age is divided by 5, the remainder is 3. When my age is divided by 7, the remainder is 1. How old am I? ("First of all, what numbers have a remainder of 1 when divided by 3? . . .") Find a binomial that multiplies by $4x - 3$ to give $16x^2 - 9$. ("Well, $16x^2 - 9$ is the difference of two squares.")

TABLE 1–4. *Building Rules to Represent Functions*

Some Related Mathematical Ideas	Several Possible Indicators	Examples
Patterns; Functions; Relations		
The NCTM Standards refer frequently to the importance of learners having consistent experience with relations, patterns, and functions. These form the content umbrella for this habit of thinking.	Uses repeating "chunks" to build a process for solving a problem ("Is there information here that let's me predict what's going to happen?") ("Am I doing the same steps over and over?")	(In the Crossing the River problem) How many one-way trips are needed if there are A adults and 2 children? ("It takes 4 one-way trips to complete the process of getting one adult across. Repeat this A times, then . . .")
Of particular relevance are input–output approaches of some curriculum materials to the concept of functions, sometimes called a function–machine approach.	Works from a table to develop a general rule and shows how it can generate any table entry ("Is there a rule or relationship here?") ("When I do the same thing with different numbers, what stays the same? What changes?")	How many Cuisenaire trains are there for any length N? ("For a smaller length m, the number of trains is . . . ")
	Describes a sequence of steps taking input to output. ("Can I write down a mechanical rule that will do this job once and for all?")	How many creases will a piece of paper have if you fold it in half n times? ("Each time I fold the paper, what does it do to the existing creases?")

TABLE 1–5. *Abstracting from Computation*

Some Related Mathematical Ideas	**Several Indicators**	**Examples**
Integers; Prime Numbers; Properties of Number Systems; Operations; Modular Arithmetic		
There are numerous opportunities before formal algebra to enable learners to abstract from calculation. Young learners are developing this habit of thinking when they are using the fact that multiplication among whole numbers is repeated addition. Later, they may internalize the distributive property in the context of an investigation when one group of students develops a rule, "Add the two numbers and multiply by 3," while another group develops the (equally valid) rule, "Multiply each of the two numbers by 3, then add them." Or, they may wonder whether, when a sum of integers is divided by an integer (call it m), the remainder is the sum of the remainders when each of the integers is divided by m.	Generalizes, using the relationships among addition, multiplication, subtraction, and division. ("How does this expression behave like that one?")	What whole numbers can be expressed as the sum of five consecutive whole numbers? ("I did it the same way I answered the question about three consecutive numbers. In this case, the number has to be divisible by 5. Take the middle of the five consecutive numbers. The original number is going to be 5 times this, because . . .")
	Uses knowledge of properties to develop shortcuts ("How can I predict without doing all the work?") ("What are my operation shortcut options?")	Compute $1 + 2 + 3 + \ldots + 98 + 99 + 100$ ("I can group in pairs that equal 101: $[1 + 100] + [2 + 99] + [3 + 98] \ldots$")
		Which is greater, 5% of 7 billion dollars or 7% of 5 billion dollars? ("I know that 5% of 7 billion dollars is the same amount as 7% of 5 billion dollars, without computing.")
	Calculates readily with letter symbols, freed from the need to know what the symbols stand for ("How does this expression behave like that expression?") ("What does this have to do with the rules of calculation?")	Investigate and explain this pattern: $1/2 - 1/3 = 1/2 \times 1/3$ $1/3 - 1/4 = 1/3 \times 1/4$ $1/4 - 1/5 = 1/4 \times 1/5$, etc. ("$1/n - 1/[n + 1] = [1/n] \times [1/(n + 1)]$ is true because . . .")
	Using equivalent expressions, simplifying or complicating, depending on need ("What other ways are there to write that expression that will bring out hidden meaning?")	$x^2 - 4 - y^2 + 4$ is simplified to $x^2 - y^2$; alternatively, for certain purposes, $x^2 - y^2$ can be "complicated" to $x^2 - 4 - y^2 + 4$.
	Computes in different systems ("How is this calculating situation like/unlike that one?")	$x^2 + 7x = 0$ has four solutions in the system of integers modulo 10.

References and Further Reading

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